# Gaskinetics and gasdynamics of orifice flow

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The paper gives the result of a study on the efflux of gases through circular apertures. The problem is considered as an example of a transition from the gasdynamic to the gaskinetic régime.

The mass flow of helium, argon and nitrogen was measured for a range of upstream pressures corresponding to (mean free path)/(aperture diameter) from about 50 to $5 \times 10^{-3}$; within this range the transition from molecular effusion to inviscid, transonic flow takes place. The theory for the two asymptotic limits is discussed and first-order corrections to the free molecular and inviscid limit formulae are given.

## 1. Introduction

Consider steady gas flow from one large container to another through a circular aperture of negligible lip thickness. The symbols $p_1$, $\rho_1$, $T_1$, etc., denote the variables of state in the upstream container, and $p_2$, $\rho_2$, $T_2$ the corresponding variables downstream, both measured at a sufficient distance from the aperture.

This is called 'orifice' flow here. For upstream densities such that the mean free path $\Lambda$ is large compared to the aperture diameter $D$, the flow corresponds to Knudsen's 'effusion' (Knudsen 1909); for sufficiently high pressure the flow approaches the potential problem of outflow of a fluid from a vessel, first studied by Kirchhoff for incompressible flow, and by Chaplygin (1904), Frankl (1947) and Guderley (1957) for compressible-fluid flow.

The study of orifice flow presented here has its origin in an attempt to find one simple and experimentally realizable flow problem for which both the Euler limit at infinite Reynolds number and the free molecular limit at zero Reynolds number are well defined and, at least in principle, theoretically understood, if not worked out in detail. In studying such a problem it is hoped to make some step in the direction of a better understanding of the transition from gasdynamic to gaskinetic flow. There do not exist many flow problems which fall within this category. Usually the free molecular limit involves unknown or only empirically known surface interaction parameters like slip and accommodation coefficients.

In this paper we will discuss only the mass flow $\dot{m}$ through a circular aperture for the case of large pressure ratios across it. To define the range of parameters and the relation of the present investigation to Knudsen's classical work and the various theoretical and experimental studies of orifice flow at high Reynolds number, it is most convenient to present all possible orifice flow régimes in a suitably defined Mach number *vs* Reynolds number plane. From the six variables $p_1, \rho_1, T_1, p_2, \rho_2, T_2$, one can eliminate three by the equation of state and by simple thermodynamics, which yields $T_1 = T_2$ since we deal with the Joule–Thomson process of a perfect gas. Choosing the three variables $p_1, \rho_1$ and $p_2$, we can define a characteristic velocity $W$ by

$$W = \left(\frac{p_1 - p_2}{\rho_1}\right)^{\frac{1}{2}},$$

and consequently Mach and Reynolds numbers by

$$Ma = \frac{W}{(p_1/\rho_1)^{\frac{1}{2}}} = \left(\frac{p_1 - p_2}{p_1}\right)^{\frac{1}{2}},$$

$$Re = \frac{WD}{\nu_1} = \frac{D}{\nu_1}\left(\frac{p_1 - p_2}{\rho_1}\right)^{\frac{1}{2}}.$$

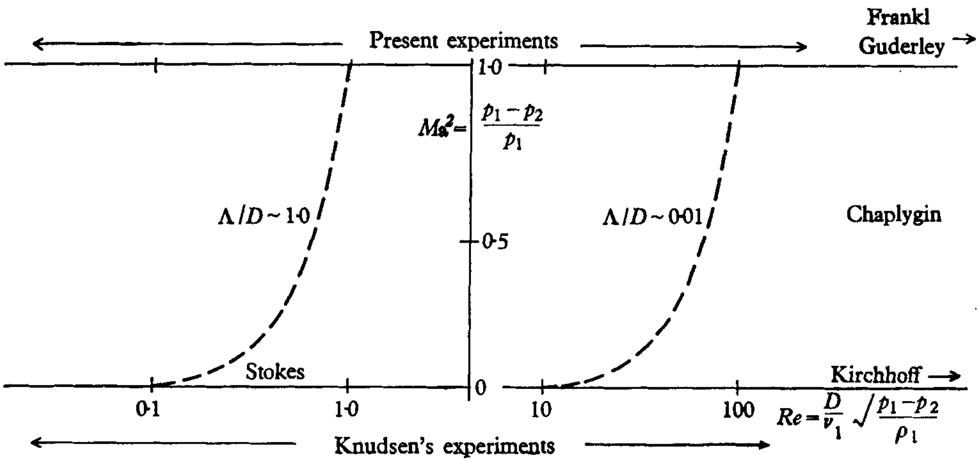


FIGURE 1. Plane of (Mach number)$^2$ *vs* Reynolds number for orifice flow.

Here $\nu_1$ is the kinematic viscosity of the gas evaluated at the equilibrium upstream conditions. It is often more convenient to use $Ma$ and $Re$ rather than the Knudsen number $\Lambda/D$ because one does not have to adopt a specific $\Lambda$ as related to $\nu$ and because $Ma$ and $Re$ are better suited to the high-pressure limit; however, one may of course replace $Ma/Re$ by $\Lambda/D$ if one prefers this notation.

$Ma$ and $Re$ can be chosen independently in the experimental arrangement. In a ($Ma^2$, $Re$)-plane (figure 1), all possible (adiabatic) orifice flows are contained within a strip $0 \leqslant Ma^2 \leqslant 1$; $0 \leqslant Re < \infty$. $Re = 0$ is the free molecular limit, $Re = \infty$ the Euler limit. Two paths between these limits are particularly distinct. For pressure ratios near unity, i.e. $p_1/p_2 \doteqdot 1$ and thus $Ma \doteqdot 0$, one passes from Knudsen flow at $Re = 0$ through a Stokes-type régime to Kirchhoff flow at $Re = \infty$. For large pressure ratios, i.e. $p_1/p_2 \gg 1$, $Ma \doteqdot 1$, one passes from free molecular flow through a region of viscous compressible fluid flow to transonic orifice flow at $Re = \infty$. Knudsen's investigations followed the paths $Ma \doteqdot 0$; the present experiments cover the range for $Ma \doteqdot 1$.

The mass flow through the orifice can be written in terms of the orifice area $A$ and the characteristic velocity $W$ simply in the form

$$\dot{m} = \Gamma \rho_1 W A,$$

where $\Gamma$ is a dimensionless factor which depends on the dimensionless parameters $Ma$, $Re$, the ratio of specific heats $\gamma$, and in the transition region possibly on slip, accommodation coefficients, and possibly a Reynolds number based on the bulk viscosity coefficient. The functional form of $\Gamma$ and hence of $\dot{m}$ in the various limits can then be deduced.

More than fifty years ago Knudsen (1909) derived his 'effusion' formula from the kinetic theory of gases; in the present notation Knudsen's result becomes

$$\Gamma_{Re\to 0} = \frac{Ma}{\sqrt{(2\pi)}}.$$

The numerical factor here is a direct (but unfortunately not very sensitive) consequence of the analytical form of the Maxwell distribution. Knudsen's experiments which did check this numerical factor were therefore at the time considered crucial in verifying the existence of a Maxwell distribution in a gas.

In the opposite limit where $Re \to \infty$, one has

$$\Gamma_{Re\to\infty} = f(\gamma, Ma);$$

in particular for the conditions with $Ma \doteqdot 1$ which are of prime interest in this paper, $\Gamma$ can be written

$$\Gamma_{Re\to\infty} = f(\gamma, 1) = \alpha(\gamma)\left[\gamma\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\right]^{\frac{1}{2}},$$

where $\alpha(\gamma)$ is a theoretically well-defined coefficient which accounts for the difference in mass flow between a smooth nozzle ($\alpha = 1$) and an orifice ($\alpha < 1$). Many experiments have been performed on orifice flow at high Reynolds numbers because of its technical importance in metering problems. Curiously enough, no experimental work on the orifice flow as an interesting and challenging problem in transonic aerodynamics seems to have been carried out. In the engineering tests the transonic character of the flow has not been appreciated, e.g. the large pressure ratios necessary to obtain maximum mass flow come as a surprise to many workers in the field.

In the present paper mass-flow measurements at pressure ratios of about $10^3$ are discussed. The measurements cover a range of Reynolds numbers from about $5\times 10^{-2}$ to $5\times 10^{2}$, or $\Lambda/D$ ratios from 50 to $5\times 10^{-3}$. In this range the transition from free molecular flow to essentially inviscid flow is completed. Some of the results have been discussed before (Liepmann 1959, 1960). Further work for small pressure ratios is under way.

## 2. The free molecular limit

Choose a system of co-ordinates such that the wall with the orifice lies in the $(y, z)$-plane. The number $n$ of molecules striking an area $A$ of the *wall* per unit time is then given by

$$n = NA\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uf(u,v,w)\,du\,dv\,dw, \tag{1}$$

where $f$ denotes the normalized equilibrium distribution function and $N$ the number of molecules per unit volume. The same type of formula gives the number of molecules passing through an aperture of area $A$, except that $f$ is now a non-equilibrium distribution function. Knudsen's effusion formula is based on the assumption that in the limit of large mean free path $\Lambda$ the lack of reflected molecules from the orifice area has a negligible effect on the distribution function, and hence the number $n$ of molecules passing through $A$ is obtained from (1) using the equilibrium distribution function. Thus one obtains the well-known expression

$$n = \tfrac{1}{4}\bar{c}_1 N_1 A \tag{2}$$

in terms of the number of molecules per unit volume $N_1$ and the mean molecular velocity $\bar{c}_1 = \sqrt{[(8/\pi) RT_1]}$ far upstream of the orifice. The mass flow is then given by

$$\dot{m} = \frac{1}{\sqrt{(2\pi)}}\rho_1 A \sqrt{(RT_1)},$$

or

$$\Gamma_{Re\to 0} = \frac{1}{\sqrt{(2\pi)}}. \tag{3}$$

To assess the validity of this limit and to estimate the degree of approximation for finite $\Lambda/D$, one has to study the nearly free molecular flow, i.e. the flow for large but finite $\Lambda/D$.

This can be carried out in a systematic fashion by iteratively solving the Boltzmann equation in order to obtain the non-equilibrium $f$, which then yields the number of passing molecules or the mass flow from (1). This has been done by Narasimha (1960). The computations are quite involved and require a number of approximations. A less systematic but simpler demonstration of the approach to free molecular flow is the following.

Following Present (1958) one can obtain Knudsen's formula (equation (2) above) in a slightly more refined way by computing the number of molecules scattered from a volume element $dV$ in the direction of $dA$ which do not suffer a collision between $dV$ and $dA$. Let $\epsilon$ denote the mean number of collisions of a molecule per unit length travelled, i.e.

$$\epsilon \equiv \Lambda^{-1} = \sqrt{2}QN,$$

where $Q$ is the collision cross-section. The number of molecules scattered per unit time in $dV$ is thus equal to $N\epsilon\bar{c}\,dV$. In the free molecular limit, $N = N_1$ everywhere and hence the number of molecules scattered such as to pass $dA$ without further collision is given by

$$dn = \int_V N_1 \epsilon_1 \bar{c}_1 \frac{dA \cos\theta}{4\pi r^2} \exp(-\epsilon_1 r)\, dV,$$

where $r$ is the distance from the volume element $dV$ to $dA$ and $\theta$ is the angle made with the normal to $dA$. Integration then gives

$$dn = \tfrac{1}{4}\bar{c}_1 N_1 dA.$$

The primary effect of the aperture is the reduction of the number of molecules in the neighbourhood of the opening due to the lack of reflexions. Thus $N$ varies from $\frac{1}{2}N_1$ in the plane of the orifice to $N_1$ at infinity. Consequently, $\epsilon$ depends on $r$ and the computation of $dn$ has to be slightly generalized to yield

$$dn = dA\,\frac{1}{4}\int_0^\infty N\epsilon\bar{c}\exp\left\{-\int_0^r \epsilon\,dr'\right\}dr. \tag{4}$$

Actually $\bar{c}$ will depend on $r$ as well (due to the predominant loss of fast molecules $\bar{c} < \bar{c}_1$) but this effect is small and will be neglected. To evaluate $dn$, let $N = \frac{1}{2}N_1$ for $r < \frac{1}{2}D$ and $N = N_1$ for $r > \frac{1}{2}D$, $\bar{c} = \bar{c}_1$. Then

$$dn = \tfrac{1}{4}N_1\bar{c}_1(1+\tfrac{1}{8}\epsilon_1 D + \ldots)\,dA.$$

For large $\Lambda/D$, i.e. small values of $\epsilon D$, this formula applies approximately to a finite orifice area $A$, and hence

$$\dot{m} = \tfrac{1}{4}\rho_1\bar{c}_1 A\left(1+\frac{D}{8\Lambda}\pm\ldots\right). \tag{5}$$

Knudsen's limit is thus justified, but the approach of $\Gamma$ to the limit is quite slow since

$$\Gamma_{Re\to 0} = \frac{1}{\sqrt{(2\pi)}}\left(1+\frac{D}{8\Lambda}\pm\ldots\right).$$

The numerical factor of $\frac{1}{8}$ agrees very well with Narasimha's computation which gave 0·13.

The increase in mass flow with increasing $D/\Lambda$ in equation (5) is due to the increase in mean free path near the orifice. For still larger $D/\Lambda$ the decrease in the density and the decrease in the temperature—and hence in $\bar{c}$—will tend to check the mass flow increase. Qualitatively the process will continue toward the isentropic conditions at very large $D/\Lambda$. But to follow the process in detail beyond the first-order terms soon becomes hopelessly complicated.

## 3. The Euler limit

We now consider the flow at infinite Reynolds number and large pressure ratio $p_1/p_2$.

For inviscid flow there exists a definite critical pressure ratio $(p_1/p_2)_c$ beyond which the downstream conditions cease to influence the upstream conditions. The maximum mass-flow is thus reached for $(p_1/p_2)_c$ for given upstream conditions. For a smooth nozzle critical pressure ratio and maximum mass flow are

$$\left(\frac{p_1}{p_2}\right)_c = \left(\frac{\gamma+1}{2}\right)^{\gamma/(\gamma-1)},$$

$$\dot{m} = \rho^* a^* A = \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)}\rho_1 A(\gamma R T_1)^{\frac{1}{2}},$$

i.e.

$$\Gamma_{Re\to\infty} = \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)}\gamma^{\frac{1}{2}}.$$

The sonic line in a sharp-edged orifice is S-shaped, and hence in the plane of the orifice the flow is partially subsonic and partially supersonic. Since the largest mass flow density occurs at local sonic velocity, the mass flow through an orifice must be less than through a comparable smooth nozzle for which the sonic line is straight. Thus we can write

$$\Gamma_{Re\to\infty} = \alpha\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)}\gamma^{\frac{1}{2}}$$

with $\alpha = \alpha(\gamma)$.

The two-dimensional problem, i.e. the flow through a slit, is sketched in figure 2 following Guderley (1957). The problem is a typical transonic one and, when transformed into the hodograph plane, leads to a Tricomi boundary-value problem. The flow becomes sonic at the lip of the slit and expands around the sharp

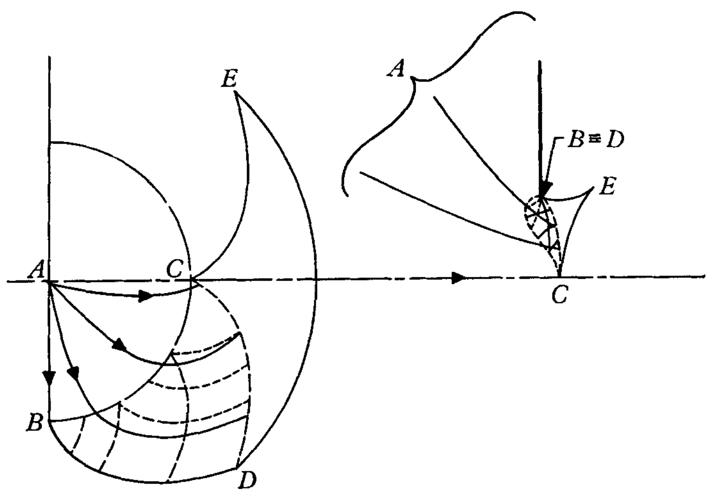


FIGURE 2. Sketches of hodograph and physical planes for the two-dimensional problem of flow through a slit (after Guderley).

edge locally like Prandtl–Meyer flow. The streamline thus maps on a characteristic in the hodograph plane. There exists a 'last characteristic' such that the flow downstream of this characteristic has no influence on the upstream conditions. This last characteristic connects the sonic point on the axis of the nozzle with the corner. In the hodograph plane this characteristic is an epicycloid from the sonic circle on the $u$-axis. Due to the symmetry in the pattern this characteristic meets the corner streamline or characteristic at $\theta = 45°$. Consequently the critical pressure ratio $p_1/p_2$ necessary for maximum mass flow is the one for which the flow past this last characteristic is established, i.e. the critical ratio is equal to the pressure ratio for a 45° turn from sonic in Prandtl–Meyer flow. This fact, though well known to the experts on transonic flow, has apparently not been generally recognized. Table 1 gives some typical values for the critical local Mach number and the critical pressure ratio; they are interesting mainly because of their large values compared with Laval nozzle flow.

The computation of the corresponding mass flow $\dot{m}$ is much more difficult, the Tricomi problem for the full hodograph equation (i.e. not the transonic approximation) has to be solved. Only one numerical computation seems to have been made: Frankl (1947) computed the flow through a two-dimensional slit for a gas with $\gamma = 1{\cdot}40$. He found $\alpha = 0{\cdot}85$.

The axisymmetrical flow corresponding to the circular orifice is even more difficult to compute because the hodograph transformation here does not lead to linear equations. Hence even the characteristics of the equation in the hodograph plane are not known *a priori*, so that the simple and general statement for the critical pressure ratio given above does not apply. One can give some simple qualitative arguments to show that the critical pressure ratios are even larger in axisymmetrical flow. Hence one expects smaller mass flow; but both should differ but little from the corresponding values for a slit.

| $\gamma$ | $Ma_c$ | $\left(\frac{p_1}{p_2}\right)_c$ |
|---|---|---|
| 5/3 | 3·50 | 58·3 |
| 7/5 | 2·76 | 25·6 |
| 4/3 | 2·63 | 21·5 |

TABLE 1

First of all note that the flow near the sharp lip is locally two-dimensional. Hence the streamline here maps again on the epicycloid in the hodograph plane. Thus the critical pressure ratio is smaller or larger than in the planar case depending on whether the characteristic from the axis intersects the epicycloid at a point corresponding to $\theta \lesseqgtr 45°$.

Now near the axis the transonic approximation to the flow equation is adequate. Let $u$ denote the excess velocity beyond $a^*$, $\theta$ the streamline angle, and $r$ the radial co-ordinate; $u$ and $\theta$ are related along the characteristic by

$$\frac{d(r\theta)}{du} = r\sqrt{(\gamma+1)}\sqrt{\left(\frac{u}{a^{*3}}\right)}.$$

For plane flow

$$\frac{d\theta}{du} = \sqrt{(\gamma+1)}\sqrt{\left(\frac{u}{a^{*3}}\right)},$$

thus

$$\frac{d\theta}{du} = \left(\frac{d\theta}{du}\right)_{\text{planar}} - \frac{\theta}{r}\frac{dr}{du};$$

near the axis $dr/du > 0$, and hence

$$\left(\frac{d\theta}{du}\right)_{\text{axisym.}} < \left(\frac{d\theta}{du}\right)_{\text{planar}}$$

Furthermore, at the intersection point of the characteristic with the epicycloid, i.e. at the corner, the flow is locally two-dimensional and hence the characteristic there approaches the slope of an epicycloid through the point. Hence the 'last characteristic' for the axisymmetrical flow intersects the epicycloid corresponding to Prandtl–Meyer flow at the corner at an angle $\theta > 45°$ and hence $(p_1/p_2)_c$ and $Ma_c$ are both larger than the corresponding two-dimensional values; one can thus expect $\dot{m}$ to be smaller for the orifice than the corresponding value for the slit.

The difference should be very slight: the difference in characteristic slope can be estimated from the transonic nozzle expansion near the axis and is small;

hence considering that even a drastic change from a critical Mach number of $Ma_c \doteqdot 1$ for nozzle flow to a critical Mach number of about 2·75 in two-dimensional orifice flow reduces $\alpha$ only from unity to 0·85, one has reason to believe that the circular orifice will have nearly the same orifice coefficient as the slit. The difference lies probably within the error limits of both the numerical computation and the experiments.

For large but finite Reynolds numbers, boundary-layer effects have to be considered. Along the orifice lip a viscous layer will exist and continue into the orifice, thus reducing the effective orifice area. From boundary-layer similarity one expects a mass-flow formula of the type

$$\dot{m} = \dot{m}_{Re\to\infty}(1 - CRe^{-\frac{1}{2}}),$$

where $C$ is some constant. A crude estimate of this constant can be obtained either by reducing the axisymmetrical problem to a Falkner–Skan flow using Mangler's transformation and approximating to the outer flow (for example by three-dimensional sink flow) or by applying—since the flow is accelerating—a simple method of the Kármán–Pohlhausen type, e.g. Walz's method as extended by Rott & Crabtree (1952) using the pressure distribution near the lip as computed by Frankl for slit flow.

A few computations like this have been carried out. However, near the lip pressure gradients are large and the actual flow is complicated by interaction effects between outer flow and boundary layer, and there does not seem to be much point in giving all the computational details. Experimentally the constant in the formula is of the order of $\frac{1}{2}$; the limiting formula for $\Gamma$ at high Reynolds number is thus roughly

$$\Gamma = \Gamma_{Re\to\infty}\left(1 - \frac{1}{2\sqrt{Re}} \pm \dots\right).$$

The complications near the lip were emphasized by the experiments, which indicated a maximum, i.e. an overshoot, in the approach to the asymptotic value at high $Re$. Such an overshoot is indeed conceivable since the boundary layer not only restricts the area but also rounds off the lip, thus tending to increase the mass flow. However, neither the experiments nor the theoretical considerations are sufficiently definite to pursue this point as yet.

## 4. Free molecular effusion

It is of interest to collect here a few simple results valid for free molecular effusion, i.e. in the limit $D/\Lambda \to 0$. We use a cylindrical system of co-ordinates in velocity space with $w$ the velocity in the plane of the orifice and $u$ normal to it. The normalized local velocity distribution of the effusing particles can then be written

$$f(u,w)\,du\,dw = 4\beta^2 uw\, e^{-\beta(u^2+w^2)}\,du\,dw,$$

with $\beta = \frac{1}{2}(RT_1)^{-1}$. The mean normal velocity and transport of momentum through the hole are then

$$\bar{u} = \sqrt{(\tfrac{1}{2}\pi RT_1)}$$

and

$$\dot{m}\bar{u} = A\rho_1\sqrt{\left(\frac{RT_1}{2\pi}\right)}\sqrt{\left(\frac{\pi RT_1}{2}\right)} = \tfrac{1}{2}p_1 A.$$

The latter result is obvious since $p_1 A$ is the net rate of change of momentum at an area $A$ of a wall; thus without reflecting the molecules back, momentum $\frac{1}{2}p_1 A$ passes through $A$ per unit time.

The flux of kinetic energy becomes

$$\dot{m}\frac{\overline{u^2+w^2}}{2} = \dot{m}2RT_1,$$

i.e. one obtains the well-known result that the flux of kinetic energy through the orifice is larger by $\frac{1}{2}RT_1$ than the product of mass flow and mean kinetic energy of the gas in the container.

The result has an interesting consequence on the energy balance. In the steady state, the mass flow $\dot{m}$ is continuously replaced. Consequently, if the effusion formula is taken literally, there appears to be a net loss of energy from the container since excess energy $\frac{1}{2}RT_1$ flows out through the orifice. Consequently the temperature in the effusing jet must differ from $T_1$ and a heat flow $q$ must exist in the container. One can work out the heat flow approximately by matching a continuum heat conduction solution for $r > \Lambda$ to the heat flux $\dot{m}2RT_1$ at a half sphere with $r = \Lambda$. One finds that the difference in temperature is of order $(D/\Lambda)^2$ and hence negligible in the limit. Furthermore, one can estimate the entropy production in the upstream container and show that it too is negligible, in agreement with more general considerations given below.

The momentum and the excess kinetic energy are imparted to the walls in the downstream container by direct wall collisions. For example, if the beam effuses into a cylindrical pipe of radius $R_0$ with $\Lambda > R_0 \gg D$, one obtains a shear stress distribution $\tau(x)$ along the wall

$$\tau(x) = \frac{p_1 A}{4\pi R_0}\frac{3x^2R_0^2}{(x^2+R_0^2)^{\frac{5}{2}}},$$

with a maximum stress at $x = \sqrt{(\frac{2}{3})}R_0$ and, of course, the integral of $\tau$ with respect to $x$ from 0 to $\infty$ is $\frac{1}{2}p_1 A$. The heat flux $q$ due to the excess beam energy is distributed similarly. Half the total number of effusing molecules collide with the wall for $x < R_0$. Consequently a density gradient along the axis of the downstream container is built up and the flow is similar to, for instance, Knudsen flow through a long tube. Note that in the effusion case the far downstream pressure $p_2$ and the mass flow $\dot{m}$ are given by the experimental conditions. The pressure $p_2'$ just downstream of the orifice is larger than $p_2$. If $p_1/p_2' \gg 1$ is required, then $R_0$ has to be chosen sufficiently large.

## 5. Entropy production in orifice flow

The total change of entropy for the complete flow from the equilibrium state (1) to the equilibrium state (2) is given by

$$\frac{S_2-S_1}{R} = \log\frac{p_1}{p_2},$$

where $S$ denotes the entropy per unit mass and $R$ the gas constant per unit mass.

If $p_1 \gg p_2$, the entropy increase per unit mass is large, but the gas flow upstream of the orifice is independent of $p_2$. Consequently the entropy increase has to be produced downstream of the orifice.

In the limit $Re \rightarrow \infty$, the flow is isentropic upstream of a shock wave and the macroscopic mechanism for entropy production is the shock wave system in the supersonic jet downstream of the orifice. For example, with the pressure ratio of $10^3$ used here, a normal shock at a Mach number somewhat larger than 10 suffices.

In the opposite limit $Re \rightarrow 0$, the problem is more interesting. From the general thermodynamic argument given above, the irreversible entropy production upstream of the orifice must be negligibly small. Consequently, the molecular distribution function must be almost everywhere locally Maxwellian. This is true for the limiting distribution in the effusing beam which can be transformed into a locally Maxwellian distribution function. It must also be true for the next approximations valid at finite Reynolds numbers.

The entropy in free molecular flow is produced downstream by the irreversible processes in the pipe flow, e.g. as in Knudsen flow through a long tube.

## 6. Apparatus and measurements

### (i) *General technique*

To obtain the mass flow $\dot{m}$, one can follow essentially two different approaches. (1) One measures the decay of the pressure in the upstream container as function of time and obtains $\dot{m}$ by differentiating the resulting decay curve. (2) One can operate at steady-state conditions, i.e. by continuously pumping the gas through the orifice and leaking a metered quantity of gas into the upstream container. When the pressure in the upstream container has reached a stationary value, the metered gas quantity equals the mass flow through the orifice at this pressure.

In the present measurements the second method was applied. The general particulars of the apparatus are dictated by the choice of the orifice diameter $D$. Here one has to compromise between the need to make $D$ large in order to reduce the corrections for finite lip thickness (the so-called Clausing correction (1932)) and to insure a nearly perfect circular hole and the limitations due to the available pumps. The ultimate vacuum and the pumping speed must be such that one can operate at large $\Lambda/D$ still maintaining a large pressure ratio across the aperture.

The pressure ratio was chosen to be $10^3$ or higher. With the ultimate vacuum of a good standard diffusion pump of approximately $5 \times 10^{-7}$ mm of mercury, one has to limit the upstream pressure to the micron range. The mean free paths for the three gases investigated, He, A and $N_2$, are at $1\,\mu$ pressure 12·7, 4·50 and 4·44 cm respectively. The orifice diameter was thus chosen to be about 0·1 cm. The dimensions of the upstream container have to be large compared to $D$ and $\Lambda$. The dimensions of the downstream container are limited only by the conditions that the number of back-scattered molecules passing through the orifice should be negligibly small. Both the upstream and downstream conditions are met by using a cylindrical vessel of about 40 cm internal diameter. Originally it was intended to use glass tubing. However, it proved impossible to obtain tubing with a wall thickness uniform enough and sufficiently free of stresses for safe operations. Hence the tank was made of steel.

The general layout of the apparatus is shown in figure 3 which is largely self-explanatory. All pressures were measured with McLeod gauges. For low pressures a cathethometer was used to read the gauge. A Pirani gauge was used only to monitor the pressure variation in the tank and to check on the attainment of a

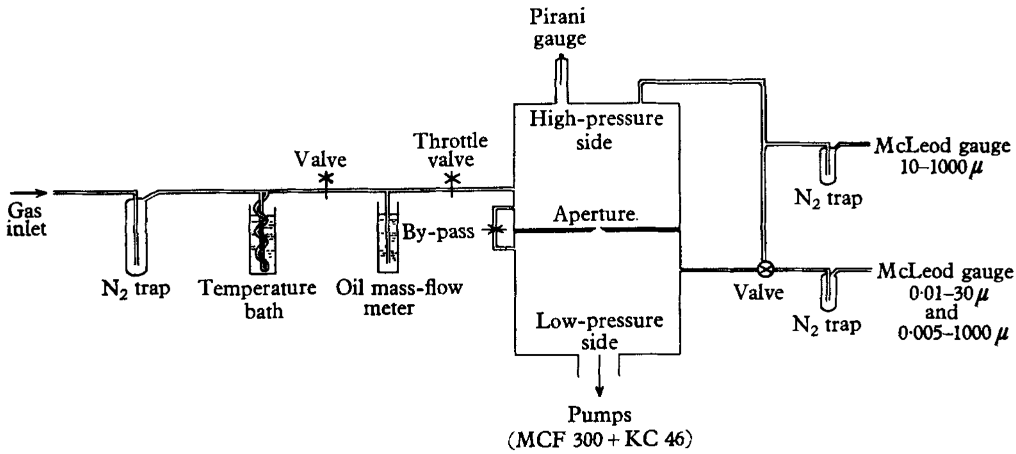


FIGURE 3. Apparatus for effusion studies.

constant pressure, i.e. steady conditions. The large volumes and the relatively small mass flows made the time constant of the system large. This had two unpleasant consequences: the time needed to measure a simple point was on average about 1 hr, and the system became quite sensitive to temperature changes in the room. The maximum permissible rate of temperature change was about 1 °C/hr.

### (ii) *Detailed construction and techniques*

*Orifice construction*

Two orifices were used. One was a circular hole drilled into a piece of 0·0025 cm thick shim stock and cemented on a circular plate with a hole of about 1 cm diameter with tapered edges. This plate in turn fitted with *O*-ring seals into the dividing wall in the tank. The orifice was selected by inspection under a large optical comparator from a large number of similarly made samples. The orifice was very nearly circular with well-defined edges and a diameter $D = 0{\cdot}1047$ cm.

This orifice construction had the disadvantage that it was impossible to check the cemented section for leaks. Hence a second orifice of integral construction was built with roughly the same dimensions: lip thickness 0·0025 cm and a diameter $D = 0{\cdot}09855$ cm. Measurements were made with both orifices; within the experimental scatter no difference was found.

*Seals, leak rate, degassing*

Steel and glass construction was used throughout. No seals except *O*-ring seals or glass-blown seals were used. The main tank was helium-tested before use. The whole apparatus remained at pressures of a micron or less for months at a time in order to reduce the degassing rate since the system could not be baked. The ultimate leak plus degassing rate thus obtained raised the pressure in the tank from the ultimate pump pressure of $10^{-6}$ mm by approximately $0{\cdot}2\,\mu$/hr. For most measurements this rate of pressure rise was entirely negligible. The measurements at the lowest pressure, however, show an increased scatter which can be attributed to variable degassing.

*Gases, mass-flow measurements*

Most measurements were made with He, A and $N_2$. A few measurements were also made with $CO_2$. The gases used were taken from commercially available tanks cleaned only by removing condensable gases in a liquid nitrogen trap. Argon was liquified in the trap and redistilled.

To regulate the mass flow into the system a number of more or less complicated schemes were tried. Actually it turned out that a good steel needle valve proved superior over most of the more complicated devices, in particular because when such a valve was used the volume in the mass-flow measuring device could be made very small indeed. This was very important because of the need to reduce scatter by temperature variation.

The mass flow was measured by timing the displacement of a silicone oil meniscus in a calibrated burette with a precision stop watch. The volume flow at atmospheric pressure which was measured in this fashion varied from about 2 to $10^5$ cm$^3$ per hour.

*Accuracy*

Each of the plotted points is an average of at least three measurements at the same setting and represents a separate absolute measurement of the mass flow; some of the points were taken more than a year apart. The estimated error from known sources, e.g. the mass-flow measurements, temperature variation, orifice geometry, etc., is about $\pm 2\,\%$. Additional errors which are difficult to assess explicitly arose from uneven degassing at low pressure, lack of attainment of complete equilibrium, and the well-known difficulties in reading absolute pressures on a McLeod gauge with great accuracy. The over-all error is apparent from the scatter in the measured points.

With an improved design and improved techniques one could certainly improve the accuracy considerably, particularly if one restricted the measurements to a limited pressure range or flow régime.

## 7. Results of the measurements

The bulk of the mass-flow measurements is shown in figures 4–6. Figure 4 gives the over-all behaviour of $\Gamma$ normalized by $\Gamma_k$, the theoretical asymptotic value for free molecular flow, as functions of $Re^{-1}$ for three gases. The figure serves mainly to show the general trend of the transition curve as well as the similarity between the three gases with molecular weights of 4, 28 and 40 respectively.

Figure 5 shows the approach of Γ to the free molecular limit. The ratio of Γ for He and A to the limit value corrected for finite channel length is plotted versus $D/\Lambda$. The general agreement in behaviour with the approximate theoretical formula for near free molecular flow is shown.

Knudsen's measurements (1909) as usually given in text-books agree to within 2 or 3 % with the limiting value at about $D/\Lambda \leqslant 0{\cdot}1$; however, it appears that Knudsen's values have not been corrected for finite lip thickness. If the Clausing

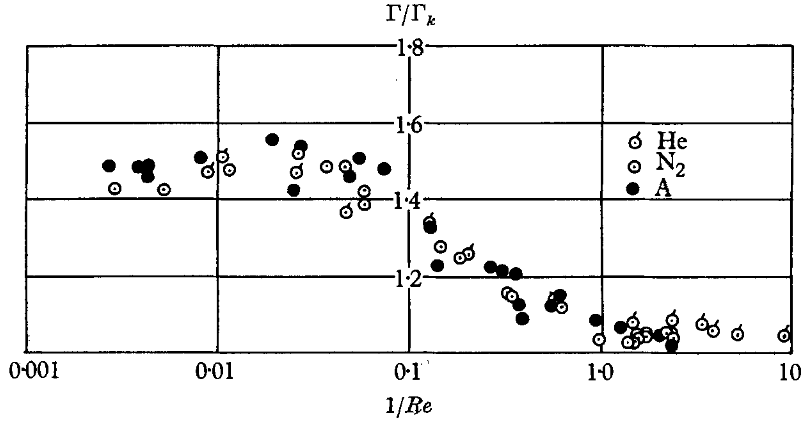


FIGURE 4. Experimental values of $\Gamma/\Gamma_k$ *vs* $1/Re$; $\Gamma_k$ is the theoretical asymptotic value of Γ for free molecular flow.

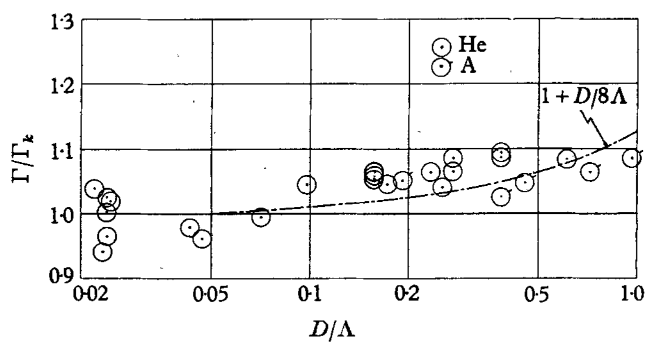


FIGURE 5. Graph showing approach of Γ to free molecular limit.

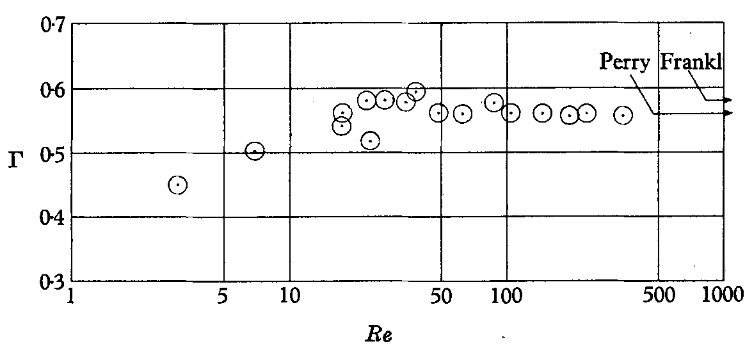


FIGURE 6. Experimental values of Γ *vs* $Re$ for $N_2$ at high pressures.

correction is applied to Knudsen's results one obtains the values in table 2. Knudsen's measured Γ's are averages of a number of measurements; the two values for $H_2$ and $O_2$ refer to the two different holes used by Knudsen which require a different correction. The Clausing correction was applied to the circle of area equal to that of the somewhat irregular holes. The corrected results are evidently internally quite consistent and in good agreement with the present measurements.

| Gas | $\Gamma/\Gamma_k$ | $\Gamma/\Gamma_k$ (corrected) |
|---|---|---|
| $H_2$ | 0·979 | 1·07 |
| | 1·02 | 1·07 |
| $O_2$ | 0·980 | 1·07 |
| | 1·04 | 1·095 |
| $CO_2$ | 0·949 | 1·04 |

TABLE 2

| Gas | $\gamma$ | $Ma_c$ | $\alpha$ (measured) | $\Gamma$ (measured) |
|---|---|---|---|---|
| A | 1·66 | 3·50 | 0·812 | 0·591 |
| $N_2$ | 1·40 | 2·76 | 0·824 | 0·564 |
| $CO_2$ | 1·30 | 2·63 | 0·830 | 0·550 |

TABLE 3

Figure 6 shows Γ for $N_2$ plotted versus *Re* at the high pressures. Since Γ for $Re \to \infty$ is not exactly known, the absolute value of Γ is plotted. The measured points approach an asymptote of $\Gamma \doteqdot 0{\cdot}56$. Frankl's theoretical value for a slit is $\Gamma = 0{\cdot}582$. Perry (1949) carried out a set of orifice measurements in air at very high Reynolds number ($\sim 10^5$). His measurements give $\Gamma = 0{\cdot}575$.

This last value should be the closest to the true theoretical limit. Still, the differences between Perry's value at high Reynolds numbers and the asymptotic value reached in the present measurements at a much lower Reynolds number are not significant in view of the experimental error limits. Even Frankl's numerical computations could conceivably be off by a few per cent, and hence the data do not yet suffice to demonstrate definitely a mass-flow difference between a slit and an equivalent orifice in transonic flow.

Finally a set of measurements at a high Reynolds number $Re \sim 200$ was made to investigate the trend of the transonic orifice flow with $\gamma$. The results are given in table 3. The measured variation of Γ and hence $\alpha$ with $\gamma$ is surprisingly small and hardly outside the experimental scatter. More accurate measurements in the transonic limit seem definitely worth while.

## 8. Conclusions

In general the measurements demonstrate the transition between the free molecular limit and the Euler limit. The transition region is relatively narrow, the Reynolds number ranges from about 0·1 to 100.

The first-order corrections valid for nearly zero and large Reynolds numbers agree with the measurements fairly well and do actually cover the larger part of the transition zone. As always, the remaining intermediate range near $Re \sim 1$ is theoretically the most difficult. The flow here should resemble compressible viscous sink flow. According to the general entropy argument, the entropy production in the flow must be small; hence dissipation and heat flow must be small or restricted to narrow zones.

The investigation has certainly posed more problems than it has solved. For example, transonic orifice flow is well worth a separate experimental study; in particular orifice flow with complex gases and reacting gases promises to be most interesting. The limiting pressure ratios and Mach numbers depend strongly on $\gamma$; for $\gamma < 5/4$, even the turning angle for expansion to zero pressure is larger than $\pi$. The theoretical conditions near the lip for this case are rather interesting. The expansion time in this restricted Prandtl–Meyer region is very short and hence relaxation effects should be observable.

In the free molecular and near free molecular régime there remain a host of problems for flows for which the wall temperature is much different from the gas temperature. This problem arises in connexion with molecular beams produced by shock-wave heated gases, which have begun to receive much attention recently. A particular problem here is the effusion of charged particles (Sturtevant 1960).

The investigations at the California Institute of Technology are being continued with the aim of studying different régimes in the $(Ma^2, Re)$-plane and in order to improve the accuracy of the measurements as well as the theoretical description.

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